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Sensitivity computations of eddy viscosity models with an application in drag computation

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SUMMARY

This paper presents a numerical study of the sensitivity of an eddy viscosity model with respect to the variation of the eddy viscosity parameter for the two-dimensional driven cavity problem and flow around a cylinder. The main objective is to provide a comparison between computing the sensitivity using sensitivity equation and computing the sensitivity using finite difference methods and also numerically illustrate the application of the sensitivity computations in improving drag flow functional.

KEY WORDS: sensitivity computations; eddy viscosity; drag computations

1. INTRODUCTION

Over the recent years, parameter sensitivity computations and analysis has become very important tools in the analysis of fluid behaviour. They describe the flow response to the variations of a parameter and therefore, from the application point of view, they provide an answer to the primary important purpose of computational fluid dynamics (CFD) analysis in determining the uncertainties arising from the choice of flow-related parameters and the accuracy of CFD predictions.

Numerical investigations are an important tool in studying fundamental aspects of turbulence and they also can be applied to generate data which can be used to formulate models for a given flow geometry. The most straightforward simulation technique is direct numerical simulation in which all turbulent flow scales are numerically resolved. Another alternative simulation technique is called large eddy simulation (LES). The idea of LES is to remove the small scales of the turbulence by means of a filtering procedure and to approximate

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the remaining scales numerically. Averaging the Navier–Stokes equations (NSE) affects the reliability of the solution. Therefore, assessing the uncertainty of the applied LES model is an important issue.

Generally speaking, sensitivity analysis of a physical system is the computation of derivatives of its state variables with respect to parameters upon which the response of the system explicitly and/or implicitly depends. There are basically two main approaches for numerically approximating the sensitivities. One is using finite differences and the other is to form an equation for the designated sensitivity and then numerically solving it. The latter is called the sensitivity equation method (SEM). SEM is classified into two different methods, the continuous sensitivity equation method (CSEM) and the automatic differentiation method (ADM). This categorization is based on obtaining the discrete sensitivity equation by first differentiating the state equation and then discretizing or first discretizing the state equation then differentiating. CSEM corresponds to 'differentiate then discretize' and ADM to 'discretize then differentiate'. In ADM, the discrete sensitivity system is obtained using automatic differentiation software. To study advantages and disadvantages of these two strategies in detail, see Reference [1].

Utilizing a flow solver code, a finite difference quotient is easy to implement. However, it may not be an efficient method for computing sensitivities (see for example Reference [2]). It produces large errors in addition to being computationally expensive in the sense that the code used for calculating a non-linear flow has to be run for two different parameter inputs at the very least (see surveys [1, 2]).

In computing the flow sensitivity via CSEM, once the flow is obtained just a linear equation needs to be solved to get the flow sensitivity. This can be done using the same program as the one used for approximating the flow. Therefore, the use of CSEM is preferable to the use of the finite difference method. A comparison between these two methods in calculating sensitivity has been presented for a specific forebody design problem by Borggaard, Gunzburger and their colleagues in Reference [2]. CSEM has been used to compute sensitivities of flows with respect to different flow-related parameters. Much work has been done on this by Borggaard, Godfrey and others (see the surveys [3–8]).

This paper examines the sensitivity of a subgrid eddy viscosity type of model to the variations of the eddy viscosity parameter. The subgrid eddy viscosity model is due to Guermont [9]. A generalization of Guermont's idea for convection diffusion problem is introduced by Layton in Reference [10]. A natural extention of Layton's model in Reference [10] to time-dependent NSE and the connection of the model to the standard formulation of a variational multiscale method is studied in Reference [11]. A formulation of this model is given below.

Call w and p the resulting approximation of the large eddy velocity and pressure and let $P' = I - P$, where P is an L^2-orthogonal projection on a subspace of $L^2(\Omega)$ [10]. Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3, be a bounded, simply connected domain with polygonal boundary $\partial\Omega$. We seek $w : \Omega \times (0, T] \to \mathbb{R}^d$ and pressure $\bar{p} : \Omega \times (0, T] \to \mathbb{R}$ satisfying,

$$
\begin{aligned}
w_t + w \cdot \nabla w - \nu \Delta w + \nabla \bar{p} - \alpha \nabla \cdot P'(\nabla w) &= f && \text{in } \Omega \times (0, T] \\
\nabla \cdot w &= 0 && \text{in } \Omega \times [0, T] \\
w &= 0 && \text{on } \partial\Omega \times [0, T] \\
w(x, 0) &= w_0(x) && \text{in } \Omega
\end{aligned}
\tag{1}
$$

Here f, the external force, is in $L^2(0,T;L^2(\Omega))$ (i.e. $[\int_0^T \int_\Omega |f(t)|^2 \,\mathrm{d}s\,\mathrm{d}t]^{1/2} < \infty$), $v > 0$ is the kinematic viscosity, which is inversely proportional to the Reynolds number Re. In this model α, the eddy viscosity parameter, is an important parameter whose values vary between 0 and 1 and is proportional to the filter length scale in LES models. Due to the fact that α in (1) causes different responses of the flow, it is natural to explore the sensitivity of the flow system and also the uncertainty of some CFD predictions which can be affected by changing the flow solution with respect to the variation of α.

Considering that the L^2-orthogonal projection P is a linear operator, using the chain rule it is easy to show that the operator P and then P' commute with differentiation with respect to α. Therefore, sensitivity of the solution (w, p) of system (1) can be computed from the following sensitivity equation, which is obtained by implicit differentiation of (1) with respect to α:

$$\begin{aligned} s_t + w \cdot \nabla s + s \cdot \nabla w - v\Delta s + \nabla q - \alpha \nabla \cdot P'(\nabla s) &= \nabla \cdot P'(\nabla w) && \text{in } \Omega \times (0, T] \\ \nabla \cdot s &= 0 && \text{in } \Omega \times [0, T] \\ s &= 0 && \text{on } \partial\Omega \times [0, T] \\ s(x, 0) &= s_0(x) && \text{in } \Omega \end{aligned} \quad (2)$$

where $s = \partial w/\partial \alpha$ and $q = \partial \bar{p}/\partial \alpha$. As it can be seen in (2), w appears in the sensitivity equation. Therefore, to complete the sensitivity analysis we need to couple (2) with (1). A complete analysis of model (1) and its sensitivity equation (2) in steady-state and time-dependent cases has been carried out in References [12–14].

This paper presents the numerical study of the flow sensitivity with respect to the eddy viscosity parameter on two experiments. We carry out the implementations on two-dimensional cases. The modifications in three dimensions are similar. Our first experiment on two-dimensional driven cavity problem focuses on providing a numerical assessment of computing the sensitivity of (1) via CSEM and forward finite difference (FFD). The purpose of the second experiment is to test the idea of using the sensitivity in improving the flow functionals, referring to the work of Anitescu and Layton in Reference [15]. This test has been performed on two-dimensional flow around a cylinder for computing the lift and drag. The test problem chosen in this experiment has been numerically studied by different groups of scientists [16, 17]. All computations are carried out using an algorithm developed from a new implicit–explicit time-stepping method introduced in Reference [18].

2. ALGORITHM

This section describes the algorithm for numerically solving Equations (1) and (2). We specifically explain how the eddy viscosity term in the model (1) and its sensitivity (2) are estimated in our calculations.

The functional spaces used in this section for w and s, $\bar{p}$ and q, $P(\nabla w)$ and $P(\nabla s)$ are defined, respectively, as follows:

$$X^h \subseteq X = H_0^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega), v = 0 \text{ on } \partial\Omega\}$$

$$Q^h \subseteq Q = L_0^1(\Omega) = \left\{ \lambda \in L^2(\Omega) : \int_\Omega \lambda \, \mathrm{d}x = 0 \right\}$$

$$L^H \subseteq L \subseteq L^2(\Omega)^{2\times 2} = \{ v = (v_{ij})_{2\times 2} : v_{ij} \in L^2(\Omega), i, j = 1, 2 \}$$

Splitting the operator P' as $I - P$ in (1) and (2), the variational formulation of these two equations in X and Q can be rewritten, respectively, as follows:

$$\begin{aligned} &(w_t, v) + (w \cdot \nabla w, v) + (v + \alpha)(\nabla w, \nabla v) - (\bar{p}, \nabla \cdot v) \\ &\quad - \alpha(P(\nabla w), P(\nabla v)) = (f, v) \quad (\lambda, \nabla \cdot w) = 0 \end{aligned} \tag{3}$$

for all $v \in X, \lambda \in Q$ and

$$\begin{aligned} &(s_t, v) + (s \cdot \nabla w + w \cdot \nabla s, v) + (v + \alpha)(\nabla s, \nabla v) - (q, \nabla \cdot v) \\ &\quad - \alpha(P(\nabla s), P(\nabla v)) = -(\nabla w, \nabla v) + (P(\nabla w), \nabla v) \quad (\lambda, \nabla \cdot s) = 0 \end{aligned} \tag{4}$$

for all $v \in X, \lambda \in Q$.

Let g denote the L^2-orthogonal projection of ∇w in (1) or (3). Then by definition of orthogonality g is obtained using an extra equation as follows:

$$\begin{aligned} &(g - \nabla w, l) = 0 \quad \forall l \in L \\ &(w_t, v) + (w \cdot \nabla w, v) + (v + \alpha)(\nabla w, \nabla v) - (\bar{p}, \nabla \cdot v) \\ &\quad + (\lambda, \nabla \cdot w) - \alpha(g, \nabla v) = (f, v) \quad \forall v \in X, \ \lambda \in Q \end{aligned} \tag{5}$$

Having w and g from (5), we consider solving the sensitivity equation using the similar idea for obtaining the L^2-orthogonal projection of ∇s as follows:

$$\begin{aligned} &(G - \nabla s, l) = 0 \quad \forall l \in L \\ &(s_t, v) + (s \cdot \nabla w + w \cdot \nabla s, v) + (v + \alpha)(\nabla s, \nabla v) - (q, \nabla \cdot v) \\ &\quad + (\lambda, \nabla \cdot s) - \alpha(G, \nabla v) = -(\nabla w, \nabla v) + (g, \nabla v) \quad \forall v \in X, \ \lambda \in Q \end{aligned} \tag{6}$$

Remark 2.1
The nature of the L^2-orthogonal projection P leads us to consider a multi-scale discretization for Equations (5) and (6) (see Reference [10]).

In the next step, we develop the time-stepping method introduced in Reference [18] and apply it to Equations in (5) and (6). The idea of the time-stepping method we consider is based on computing local and stabilizing terms implicitly and nonlocal and unstabilizing terms explicitly. In both Equations (5) and (6) the L^2-orthogonal projection P accounts for nonlocal character (i.e. its matrix has a large bandwidth). The advantage of the explicit structure for the global unstable part in the system of equations is that its action on a given vector is inexpensive to perform.

Here w_n^h and s_n^h represent $w^h(t_n)$ and $s^h(t_n)$, respectively. Let h and H denote the size of the fine and coarse mesh, respectively. Then the following fully discrete scheme is considered for implementing a code to approximate the solution of (5) and (6). The equations for the LES model are given below

$$
\begin{aligned}
&(g_n^H - \nabla w_n^h, l^H) = 0 \quad \forall l^H \in L^H \\
&\left(\frac{w_{n+1}^h - w_n^h}{\Delta t}, v^h\right) + (w_n^h \cdot \nabla w_{n+1}^h, v^h) + (v+\alpha)(\nabla w_{n+1}^h, \nabla v^h) - (\bar{p}_{n+1}^h, \nabla \cdot v^h) \\
&\quad + (\lambda^h, \nabla \cdot w_{n+1}^h) - \alpha(g_n^H, \nabla v^h) = (f_{n+1}, v^h) \quad \forall v^h \in X^h,\ \lambda^h \in Q^h
\end{aligned} \tag{7}
$$

Thus for the sensitivity, we solve

$$
\begin{aligned}
&(G_n^H - \nabla s_n^h, l^H) = 0 \quad \forall l^H \in L^H \\
&\left(\frac{s_{n+1}^h - s_n^h}{\Delta t}, v^h\right) + (s_{n+1}^h \cdot \nabla w_{n+1}^h + w_{n+1}^h \cdot \nabla s_{n+1}^h, v^h) + (v+\alpha)(\nabla s_{n+1}^h, \nabla v^h) \\
&\quad - (q_{n+1}^h, \nabla \cdot v^h) + (\lambda^h, \nabla \cdot s_{n+1}^h) - \alpha(G_n^H, \nabla v^h) \\
&\quad\quad = -(\nabla w_{n+1}^h, \nabla v^h) + (g_{n+1}^H, \nabla v^h) \quad \forall v^h \in X^h,\ \lambda^h \in Q^h
\end{aligned} \tag{8}
$$

3. SENSITIVITY COMPUTATIONS

The goal of this section is first to numerically illustrate a comparison of the sensitivity computation via two different strategies. One method uses the discretized sensitivity Equation (8), and the other uses the forward finite difference

$$\frac{w(\alpha + \Delta\alpha) - w(\alpha)}{\Delta\alpha}$$

by computing the average velocity w from (7) for two inputs $\alpha + \Delta\alpha$ and α. We also present one of the most important applications of sensitivity in identifying the reliable values of the eddy viscosity parameter α.

The numerical experiments are performed on the two-dimensional driven cavity problem [19]. The flow domain Ω is $[0,1] \times [0,1]$. The upper boundary moves with the velocity $w(t,x,y) = (16x^2(1-x^2), 0)^t$. The initial data is chosen to be $w(0,x,y) = (3y^2 - 2y, 0)^t$ in Ω. It is clear that since the initial and boundary conditions for w do not depend on α, we have zero initial and boundary conditions for the sensitivity s.

We pick X^h, Q^h and L^H to be finite element space of piecewise polynomials of degree 2, piecewise linears and piecewise constants, respectively. Sensitivity s^h is computed at each time-step t_i through computing $w^h(t_{i+1})$. All computations are carried out with $v = 0.0001$ and $h = \frac{1}{36}$ and the uniform step $\Delta t = 0.001$ for 1000 steps. The program has been implemented by FreeFem++ [20], which uses the finite element method for solving PDEs. See References [12–14] for some numerical assessments on the convergence results of (7) and (8) for the specific cavity problem.

Let s_{SEM} and s_{FFD} denote the sensitivity computation via SEM and FFD, respectively. Table I presents $\|s_{\text{SEM}}(t)\|_{L^2(\Omega)}$ and $\|s_{\text{FFD}}(t)\|_{L^2(\Omega)}$ for different values of parameter α with $\Delta\alpha = 0.0001$ at times $t = 0.01, 0.1, 1$.

It can be observed that for $\alpha \leqslant 0.025$, the computed sensitivity in both methods SEM and FFD agree for small time and are very different at the final time. As α takes larger values than 0.025, sensitivity values via SEM and FFD are close over the whole time interval.

Following Table I, you see the plot of computed sensitivity using both methods for $\alpha = 0.025$ at $t = 0.01$ and $t = 1$. Left plots are obtained from SEM and right plots from FFD. (Figures 1 and 2)

The natural way to compute the sensitivity of the average velocity with respect to the variation of parameter α is via computing $\alpha \|\partial w/\partial \alpha\|_{L^\infty(0,T;L^2)}$ (note: $\|v(t)\|_{L^\infty(0,T;L^2)} = \max_{0 \leqslant t \leqslant T} \|v(t)\|^2_{L^2(\Omega)}$). This is because of the fact that

$$\alpha \frac{\partial w}{\partial \alpha} \approx w(0) - w(\alpha) \tag{9}$$

Values of $\alpha \|s_{\text{SEM}}\|_{L^\infty(0,T;L^2)}$ and $\alpha \|s_{\text{FFD}}\|_{L^\infty(0,T;L^2)}$ for different values of parameter α for $T = 1$ are listed in Table II. This table demonstrates that the large eddy velocity w is highly sensitive

Table I. Sensitivity via SEM and FFD at different times.

α	Time	0.01	0.1	1
0.75	SEM	0.137967	0.132143	0.132229
	FFD	0.142969	0.11632	0.116324
0.5	SEM	0.194026	0.173414	0.174014
	FFD	0.205607	0.160756	0.160554
0.25	SEM	0.340816	0.301911	0.287039
	FFD	0.360893	0.298393	0.276047
0.075	SEM	0.822536	1.15329	0.661375
	FFD	0.827503	1.12457	0.649798
0.05	SEM	1.05194	1.75986	0.940254
	FFD	1.05532	1.66294	0.838625
0.025	SEM	1.54487	3.39374	75.8015
	FFD	1.55716	3.03512	1.75201
0.0075	SEM	3.46636	5.71883	767236
	FFD	3.74463	9.35573	21.767
0.005	SEM	5.41812	14.7976	1.66591e + 7
	FFD	5.78659	16.292	81.4544

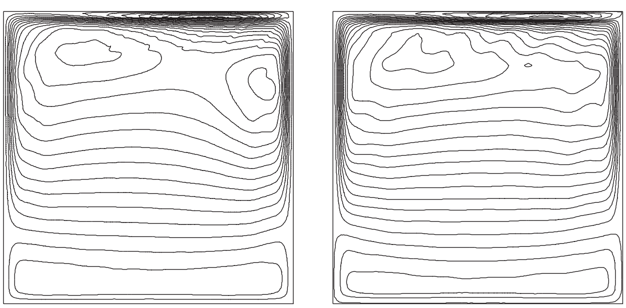

Figure 1. Similarity of sensitivity using SEM (left) and FFD (right) at $t = 0.01$.

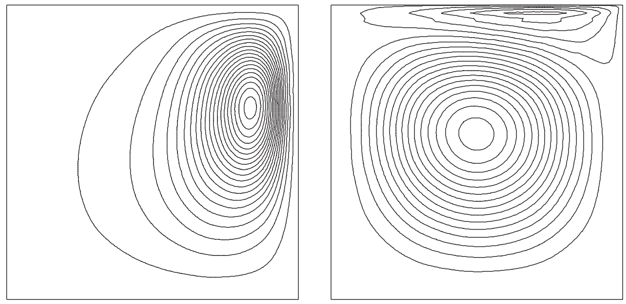

Figure 2. Difference of sensitivity computed by SEM (left) and FFD (right) at final time.

Table II. Sensitivity values via SEM and FFD for different α.

α	$\alpha\|s_{\mathrm{SEM}}\|_{L^\infty(0,T;L^2)}$	$\alpha\|s_{\mathrm{FFD}}\|_{L^\infty(0,T;L^2)}$
0.75	0.148275	0.129462
0.5	0.13029	0.18623
0.25	0.110736	0.10332
0.075	0.092992	0.088566
0.05	0.094626	0.085014
0.025	1.89503	0.08292
0.0075	5754.27	0.163719
0.005	8.32955e + 4	0.4075515

with respect to the small values of α. As α gets larger, w becomes less sensitive. Table II shows a small sensitivity using FFD overall with respect to α. Comparing the sensitivity values in Table II, sensitivities obtained from SEM are close to the ones computed by FFD for $\alpha > 0.025$. It can be observed that $\alpha\|\partial w/\partial\alpha\|_{L^\infty(0,T;L^2)}$ follows the same pattern in both methods. As α becomes large up to 0.075 then $\alpha\|\partial w/\partial\alpha\|_{L^\infty(0,T;L^2)}$ decreases and for $\alpha \geqslant 0.075$, $\alpha\|\partial w/\partial\alpha\|_{L^\infty(0,T;L^2)}$ starts to increase.

Note that (9) suggests the reliability of an approximated solution to correspond to the lower values of $\alpha\|\partial w/\partial\alpha\|_{L^\infty(0,T;L^2)}$. In Table II, the lowest sensitivity values using SEM and FFD appear to be for α in the interval $[0.05, 0.075]$ and $[0.025, 0.075]$, respectively. Thus from both approximations of sensitivity in Table II the interval $[0.05, 0.075]$ is the reliable interval for α.

Remark 3.1

In the case when the parameter α is small, the computed large eddy velocity w is a more accurate estimation and therefore it includes more scales in its structure. Ultimately, more scales are also computed in the sensitivity s and hence to get a more accurate sensitivity for small α, the sensitivity equation must be solved on a finer mesh.

4. SENSITIVITY IN DRAG COMPUTATION

In Reference [15], Anitescu and Layton proposed that the approximation of a flow functional $J(u)$ can be improved via sensitivity. This idea is based on using the linear approximation to

$J(w(0)) = J(u)$, which yields

$$J(u) \approx J(w(\alpha)) - \alpha J'(w(\alpha)) \cdot s \tag{10}$$

The major purpose of this section is to establish whether constructive conclusions in Reference [15] can be drawn from a numerical test in computing lift and drag functional for the two-dimensional flow around a cylinder. Notice that for linear functionals $J' = J$ and therefore the approximation (10) is obtained from

$$J(u, p) \approx J(w, \bar{p}) - \alpha J(s, q) = J(w - \alpha s, \bar{p} - \alpha q) \tag{11}$$

Let $u = (u_1, u_2)^t$ and $T = 4$. For the given geometry Ω shown in Figure 3, consider the following Navier–Stokes problem,

$$\begin{aligned} u_t + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0 \qquad \text{in } \Omega \times (0, T) \\ \nabla \cdot u &= 0 \qquad \text{in } \Omega \times (0, T) \\ u(0, x, y) &= (0, 0)^t \quad \text{in } \Omega \end{aligned}$$

The geometry and the boundary conditions are indicated in the following. The channel height and width is, respectively, 0.41 m and 2.2 m. A cylinder with radius 0.05 m has been centred at point $(0.2, 0.2)$.

The boundary and initial conditions are given below:

$$\begin{aligned} u_1(t, x, 0) &= u_2(t, x, 0) = 0 \\ u_1(t, x, 0.41) &= u_2(t, x, 0.41) = 0 \\ u_1(t, x, y)|_{\partial B} &= u_2(t, x, y)|_{\partial B} = 0 \\ u_1(0, x, y) &= u_2(0, x, y) = 0 \end{aligned}$$

The outflow condition is set free, i.e. u is numerically solved on the outlet boundary, and for $0 \leqslant t \leqslant 4$, the inflow condition is

$$\begin{aligned} u_1(t, 0, y) &= \frac{6}{(0.41)^2}\, y(0.41 - y) \sin\left(\frac{\pi t}{4}\right) \\ u_2(t, 0, y) &= 0 \end{aligned}$$

Figure 3. Geometry of 2D-flow around cylinder.

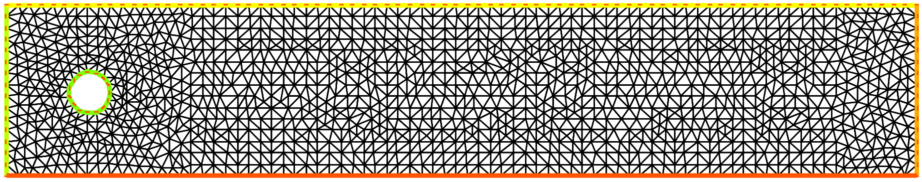

Figure 4. Mesh in a channel with size $\frac{1}{36}$ and $\frac{1}{49}$.

Table III. Reference values for drag.

ν (Re)	Drag at $T=1$	Max. drag	Drag at $T=3$	Drag at $T=4$
1 (1)	45.0804	63.7703	45.1044	0.016924
0.5 (2)	37.3295	52.8079	37.3521	0.0160216
0.25 (4)	32.708	46.2713	32.7296	0.01527
0.1 (10)	29.1195	41.1958	29.1401	0.0145559
0.01 (100)	25.494	36.0677	25.5134	0.0137471
0.001 (1000)	24.9490	35.29035	24.96835	0.01367345
0.0001 (10 000)	24.82055	35.1186	24.83985	0.01366715

The lift and drag functional for Navier–Stokes equations is given by

$$J(u,p)=\oint_B (n_x,n_y)\cdot[pI-2\nu\nabla^s u]\cdot \mathbf{a}\,\mathrm{d}s \tag{12}$$

where (n_x,n_y) denotes the normal vector on the cylinder boundary B directing into the channel, $\nabla^s u$ presents the deformation tensor and is $\frac{1}{2}(\nabla u+\nabla u^t)$, the unit vector $\mathbf{a}$ in the positive direction of x-axis or negative direction of y-axis yield the drag or lift flow functional. The reference value of drag for this test problem is obtained by performing the direct numerical solution method to a uniform mesh that is of size $\frac{1}{100}$ for each side of the channel and of size $\frac{1}{121}$ for around the cylinder (e.g. see Figure 4). Table III indicates the reference values of drag at time $T=1$, 2, 3 and 4 for different values of ν. Note that the maximum value of drag occurs at time $T=2$.

Here we approximate the values in Table III via two approaches. One is by replacing the large eddy velocity and pressure $(w,\bar{p})$ into (12) for (u,p). Secondly we use the sensitivity and compute the drag with $\bar{p}-\alpha q$ and $w-\alpha s$ for the pressure and velocity using (10) or (11).

The approximated flow functional for lift and drag using $(w,\bar{p})$ and $(w-\alpha s,\bar{p}-\alpha q)$ with parameter $\alpha=0.00125$ and a mesh of size $\frac{1}{49}$ for the sides of channel and $\frac{1}{64}$ for around the cylinder are listed in Tables IV and V. Table V indicates an improved estimation of drag for all ν at any time. This table specially presents more accurate values for $\nu<0.1$.

Looking to the sensitivity quantities for different values of parameter ν, one observes that flow is insensitive for $\nu\geqslant 0.1$. Therefore, the computed drag values using (s,q) shows a small improvement in comparison to the ones computed using $(w,\bar{p})$. According to Table VI for small values of ν (i.e $\nu<0.1$), the flow becomes more sensitive and applying sensitivity helps in obtaining a better improved values of the drag functional.

Table IV. Lift and drag approximation by large eddy velocity and pressure.

v (Re)	Drag at $T=1$	Max. drag	Drag at $T=3$	Drag at $T=4$
1 (1)	44.7951	63.3666	44.8188	0.0167302
0.5 (2)	37.0245	52.3764	37.0469	0.0158283
0.25 (4)	32.3981	45.833	32.4195	0.0150842
0.1 (10)	28.8309	40.7875	28.8513	0.0143996
0.01 (100)	25.331	35.6787	25.3505	0.0137419
0.001 (1000)	24.6588	35.0094	24.6782	0.0136678
0.0001 (10 000)	24.5272	34.6832	24.5465	0.0136627

Table V. Lift and drag approximation using sensitivities.

v (Re)	Drag at $T=1$	Max. drag	Drag at $T=3$	Drag at $T=4$
1 (1)	44.8188	63.4001	44.8425	0.0167311
0.5 (2)	37.0501	52.4125	37.0724	0.0158304
0.25 (4)	32.4274	45.8743	32.4487	0.0150879
0.1 (10)	28.8683	40.8403	28.8887	0.0144059
0.01 (100)	25.5047	36.0829	25.5242	0.0137527
0.001 (1000)	24.9592	35.3113	24.9785	0.0136751
0.0001 (10 000)	24.8339	35.134	24.8532	0.0136676

Table VI. Sensitivity for different values of v.

v	$\alpha\|w_\alpha\|_{L^\infty(0,T;L^2)}$
1	7.19057e − 06
0.5	2.25838e − 05
0.25	6.9098e − 05
0.1	0.000288244
0.01	0.00483735
0.001	0.0155576
0.0001	0.0201101

Figure 4 shows an example of a uniform mesh with size $h=\frac{1}{36}$ for the sides of the channel and $h=\frac{1}{49}$ for around the cylinder followed by Figure 5 showing the scaled velocity vector field and the computed sensitivity norm for $v=0.0001$ on the same mesh.

5. CONCLUSION

Sensitivity of the subgrid eddy viscosity model with respect to the variations of eddy viscosity parameter using two different methods, SEM and FFD, was computed with a first-order implicit–explicit time-stepping scheme for the two-dimensional driven cavity problem. The numerical comparison between SEM and FFD shows that the approximated sensitivities via

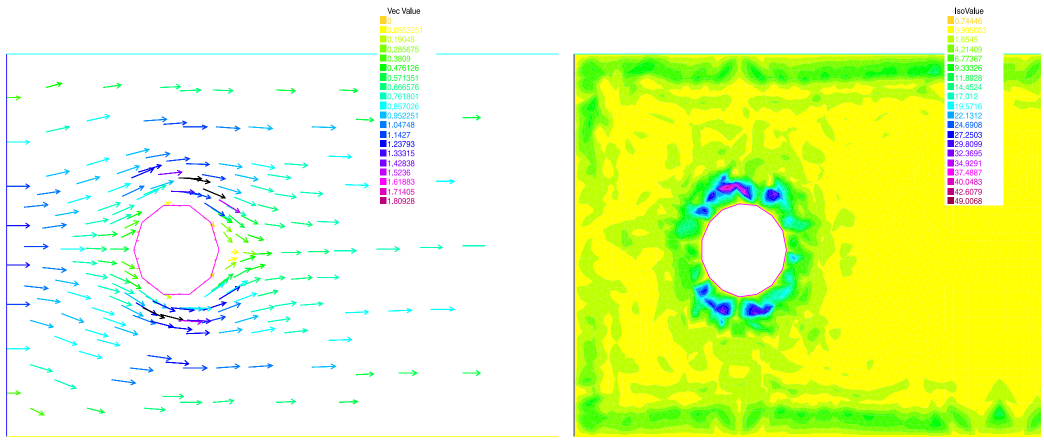

Figure 5. Velocity vector field and sensitivity norm for $\nu = 0.0001$.

these two strategies are very close in a small time interval. In addition, sensitivity computations for this experiment with different values of α suggest $[0.05, 0.075]$ as the reliable interval for values of α.

Our conclusions in the second experiment have been drawn from a numerical test in computing drag functional for the two-dimensional flow around a cylinder. This test justifies the use of sensitivity as a first-order correction term to improve the flow functional approximations.

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